

Bulk correlation functions in 2D quantum gravity¹I.K. Kostov¹ and V.B. Petkova²¹ *Service de Physique Théorique, CNRS – URA 2306,
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72 Tsarigradsko Chaussée, 1784 Sofia, Bulgaria***Abstract**

We compute bulk 3- and 4-point tachyon correlators in the 2d Liouville gravity with non-rational matter central charge $c < 1$, following and comparing two approaches. The continuous CFT approach exploits the action on the tachyons of the ground ring generators deformed by Liouville and matter “screening charges”. A by-product general formula for the matter 3-point OPE structure constants is derived. We also consider a “diagonal” CFT of 2D quantum gravity, in which the degenerate fields are restricted to the diagonal of the semi-infinite Kac table. The discrete formulation of the theory is a generalization of the *ADE* string theories, in which the target space is the semi-infinite chain of points.

1. Introduction and summary

The observation that the operator product expansions of the physical operators in the effective 2d CFT describing the quantum Liouville gravity reduce, modulo BRST commutators, to simple “fusion” relations is an old one [1]. The ghost number zero operators were argued to close a ring, the “ground ring”, which furthermore preserves the tachyon modules. It is assumed that in the rational case it coincides with the fusion ring of the minimal $c < 1$ theories, see in particular the recent work in [2]. The action of the ground ring on tachyon modules was used to derive functional recursive identities for the tachyon correlation functions [3,4,5,6].

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In this work (see [7] for a more detailed presentation) we reconsider and extend this approach for constructing bulk tachyon correlators. We study a non-rational, or quasi-rational, CFT of 2D quantum gravity, whose effective action is that of a gaussian field perturbed by both Liouville and matter “screening charges”.

We start with a direct evaluation of the 3-point function as a product of Liouville and matter OPE structure constants. For this purpose we derive an explicit expression, formula (3.3) below, for the general matter 3-point OPE structure constants. (This result was independently obtained by Al. Zamolodchikov [8], with a different normalization of the fields.) Then we write difference recurrence equations for the 4-point function of tachyons.

We also consider another non-rational CFT of 2D quantum gravity, in which the degenerate fields along the diagonal of the semi-infinite Kac table form a closed algebra. The CFT in question, which we call ‘diagonal CFT’, is described by a perturbation with the four tachyon operators whose Liouville component has dimension one and whose matter component has dimension zero.

A microscopic realization of this diagonal CFT is given by a non-rational extension of the *ADE* string theories introduced in [9] with semi-infinite discrete target space. The loop gas representation of the microscopic theory leads to a target space diagram technique [10], which allows to calculate efficiently the tachyon correlation functions [11,12]. We find agreement between the “diagonal” CFT and the discrete results.

2. Non-rational 2D gravity: effective action, local fields, ground ring

The effective action of Euclidean Liouville gravity (taken on the sphere) is a perturbation of the gaussian action

$$\mathcal{A}^{\text{free}} = \frac{1}{4\pi} \int d^2x \left[(\partial\phi)^2 + (\partial\chi)^2 + (Q\phi + ie_0\chi)\hat{R} + 4(\mathbf{b}\partial_{\bar{z}}\mathbf{c} + \bar{\mathbf{b}}\partial_z\bar{\mathbf{c}}) \right] \quad (2.1)$$

of the Liouville ϕ , and matter χ , fields and a pair of reparametrization ghosts. The background charges, $Q = \frac{1}{b} + b$ and $e_0 = \frac{1}{b} - b$ are parametrized by a real b , so that the total central charge of this conformal theory is trivial

$$c_{\text{tot}} \equiv c_L + c_M + c_{\text{ghosts}} = [13 + 6(b^2 + \frac{1}{b^2})] + [13 - 6(b^2 + \frac{1}{b^2})] - 26 = 0. \quad (2.2)$$

The physical fields, or the “on-mass-shell” tachyons in the string theory interpretation, are products of Liouville and matter vertex operators of total dimension $(1, 1)$ [13,14]

$$\frac{\gamma(1-\alpha^2+e^2)}{\pi} e^{2ie\chi} e^{2\alpha\phi} = \frac{1}{\pi} \gamma(\epsilon b^\epsilon P) e^{i(e_0-P)\chi+(Q-\epsilon P)\phi} = V_\alpha^\epsilon, \quad \epsilon = \pm 1, \quad (2.3)$$

$$e(e - e_0) + \alpha(Q - \alpha) = 1 \Rightarrow e = \alpha - b, \text{ or, } e = -\alpha + \frac{1}{b}. \quad (2.4)$$

The parameters P and ϵ in (2.3) are interpreted as the tachyon target space momentum and chirality. In the “leg factor” normalization in (2.3), $\gamma(x) = \Gamma(x)/\Gamma(1-x)$.

The BRST invariant operators associated with (2.3) are obtained either by integrating over the world sheet, or by multiplying with the ghost field $\mathbf{c}\bar{\mathbf{c}}$ of dimension $(-1, -1)$

$$\mathcal{T}_P^{(\pm)} = T_\alpha^\pm \equiv \int V_\alpha^\pm \quad \text{or} \quad \mathcal{W}_P^{(\pm)} = W_\alpha^\pm \equiv \mathbf{c}\bar{\mathbf{c}} V_\alpha^\pm. \quad (2.5)$$

In the n -point tachyon correlators $n-3$ vertex operators are integrated over the worldsheet and three are placed, as usual, at arbitrary points, say $0, 1$ and ∞ .

The ground ring of BRST invariant operators of zero dimension and zero ghost number [1] is generated by the two operators $a_\pm(x) = a_\pm(z) a_\pm(\bar{z})$,

$$\begin{aligned} a_-(z) &= (\mathbf{b}(z)\mathbf{c}(z) - b^{-1}\partial_z(\phi(z) + i\chi(z))) e^{-b(\phi(z) - i\chi(z))} \\ a_+(z) &= (\mathbf{b}(z)\mathbf{c}(z) - b\partial_z(\phi(z) - i\chi(z))) e^{-b^{-1}(\phi(z) + i\chi(z))}. \end{aligned} \quad (2.6)$$

Unlike the tachyons, the operators (2.6) are made of Liouville and matter vertex operators, both corresponding to degenerate Virasoro representations. We will consider deformations of these free field operators determined by pairs of Liouville and matter “screening charges”, i.e. by the interaction actions

$$\begin{aligned} \mathcal{A}_{\text{int}} &= \int (\mu_L e^{2b\phi} + \mu_M e^{-2ib\chi}) = \lambda_L \mathcal{T}_{e_0}^{(+)} + \lambda_M \mathcal{T}_Q^{(+)}, \\ \tilde{\mathcal{A}}_{\text{int}} &= \int (\tilde{\mu}_L e^{2\phi/b} + \tilde{\mu}_M e^{2i\chi/b}) = \tilde{\lambda}_L \mathcal{T}_{e_0}^{(-)} + \tilde{\lambda}_M \mathcal{T}_{-Q}^{(-)}. \end{aligned} \quad (2.7)$$

The renormalized by the leg factors in (2.3) coupling constants are

$$\begin{aligned} \lambda_L &= \pi\gamma(b^2)\mu, & \tilde{\lambda}_L &= \pi\gamma(\frac{1}{b^2})\tilde{\mu}, \\ \lambda_M &= \pi\gamma(-b^2)\mu_M, & \tilde{\lambda}_M &= \pi\gamma(-\frac{1}{b^2})\tilde{\mu}_M. \end{aligned} \quad (2.8)$$

The four terms in (2.7) describe perturbations that act separately on the matter and Liouville fields. In such a theory the 3-point function factorizes into a product of matter and Liouville components.

One can imagine more general perturbations by integrated tachyon fields, which affect simultaneously the matter and Liouville fields. We will study the simplest example of such a perturbation, which is described by the two Liouville “screening charges” as well as by a pair of tachyons related to the latter by a matter charge reflection $(e, \alpha) \rightarrow (e_0 - e, \alpha)$, namely $(0, b) \rightarrow (e_0, b)$ and $(0, b^{-1}) \rightarrow (e_0, b^{-1})$. This perturbation, which we call *diagonal perturbation*, is described by the action

$$\mathcal{A}_{\text{int}}^{\text{diag}} = \lambda_L (\mathcal{T}_{e_0}^{(+)} + \lambda_M \tilde{\lambda}_M \mathcal{T}_{-e_0}^{(-)}) + \tilde{\lambda}_L (\mathcal{T}_{e_0}^{(-)} + \lambda_M \tilde{\lambda}_M \mathcal{T}_{-e_0}^{(+)}). \quad (2.9)$$

3. The tachyon 3-point function as a product of Liouville and matter OPE constants

The 3-point function in the non-rational CFT of 2D gravity described by the perturbation (2.7) factorizes to a product of the matter and Liouville three-point OPE constants

$$G_3^{\varepsilon_1 \varepsilon_2 \varepsilon_3}(\alpha_1, \alpha_2, \alpha_3) = \left\langle \mathcal{W}_{P_1}^{(\varepsilon_1)} \mathcal{W}_{P_2}^{(\varepsilon_2)} \mathcal{W}_{P_3}^{(\varepsilon_3)} \right\rangle = \frac{C^{\text{Liou}}(\alpha_1, \alpha_2, \alpha_3) C^{\text{Matt}}(e_1, e_2, e_3)}{\pi^3 \prod_{j=1}^3 \gamma(\alpha_j^2 - e_j^2)}. \quad (3.1)$$

Here α_i and e_i are solutions of the on-mass-shell condition (2.4).

For the Liouville 3-point constant we take the expression derived in [15,16]:

$$C^{\text{Liou}}(\alpha_1, \alpha_2, \alpha_3) = \left(\lambda_L^{1/b} b^{2e_0} \right)^{Q-\alpha_1-\alpha_2-\alpha_3} \frac{\Upsilon_b(b) \Upsilon_b(2\alpha_1) \Upsilon_b(2\alpha_2) \Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_{123}-Q) \Upsilon_b(\alpha_{23}^1) \Upsilon_b(\alpha_{13}^2) \Upsilon_b(\alpha_{12}^3)} \quad (3.2)$$

with notation $\alpha_{12}^3 = \alpha_1 + \alpha_2 - \alpha_3$, $\alpha_{123} = \alpha_1 + \alpha_2 + \alpha_3$, etc. This constant is symmetric with respect to the duality transformation $b \rightarrow b^{-1}$, $\lambda_L \rightarrow \lambda_L^{1/b^2} = \tilde{\lambda}_L$.

For the matter 3-point OPE constant we obtain the expression

$$\begin{aligned} C^{\text{Matt}}(e_1, e_2, e_3) &= \left(\lambda_M^{1/b} b^{2Q} \right)^{e_1+e_2+e_3-e_0} \frac{\hat{\Upsilon}_b(0) \hat{\Upsilon}_b(2e_1) \hat{\Upsilon}_b(2e_2) \hat{\Upsilon}_b(2e_3)}{\hat{\Upsilon}_b(e_{123}-e_0) \hat{\Upsilon}_b(e_{23}^1) \hat{\Upsilon}_b(e_{13}^2) \hat{\Upsilon}_b(e_{12}^3)} \\ &= \frac{\lambda_L^{\frac{1}{b}(Q-\sum_i \alpha_i)} \lambda_M^{-\frac{1}{b}(e_0-\sum_i e_i)}}{b^{\sum_i \epsilon_i} \prod_{i=1}^3 \gamma(b^{\epsilon_i}(Q-2\alpha_i))} \frac{1}{C^{\text{Liou}}(\alpha_1, \alpha_2, \alpha_3)}, \quad \alpha_i = \epsilon_i e_i + b^{\epsilon_i} \end{aligned} \quad (3.3)$$

invariant under $b \rightarrow -1/b$, $\lambda_M \rightarrow (\lambda_M)^{-1/b^2} = \tilde{\lambda}_M$. The derivation of (3.3) repeats the one for the Liouville case in [17], where the formula (3.2) was determined for positive, irrational b^2 as the unique (smooth) solution of a pair of functional relations, see [7] for more details. The second line of (3.3) holds for any choice of the three signs ϵ_i using the reflection properties of the Liouville OPE constant (3.2). The function in the first line is defined as

$$\hat{\Upsilon}_b(x) := \frac{1}{\Upsilon_b(x+b)} = \frac{1}{\Upsilon_b(-x+\frac{1}{b})} = \hat{\Upsilon}_b(e_0-x) = \hat{\Upsilon}_{\frac{1}{b}}(-x) \quad (3.4)$$

and satisfies the functional relations

$$\hat{\Upsilon}_b(x-b) = \gamma(bx) b^{1-2bx} \hat{\Upsilon}_b(x), \quad \hat{\Upsilon}_b(x+\frac{1}{b}) = \gamma(-\frac{1}{b}x) b^{-1-\frac{2}{b}x} \hat{\Upsilon}_b(x). \quad (3.5)$$

Its logarithm admits an integral representation as the one for $\log \Upsilon_b$, with Q replaced by e_0 (whence invariant under the change $b \rightarrow -1/b$), which is convergent (for $b > 0$) in the strip $-b < \text{Re } x < \frac{1}{b}$. The normalization in (3.3) is fixed by the choice

$$C^{\text{Matt}}(e_1, e_2, e_3) = 1, \quad \text{for } \sum_i e_i = e_0. \quad (3.6)$$

For $\sum_i e_i - e_0 = mb - \frac{n}{b}$, n, m non-negative integers, the expression (3.3) is finite for generic b^2 and e_i and reproduces, up to the powers $(-\mu_M)^m (-\tilde{\mu}_M)^n$, the 3-point Dotsenko-Fateev constant in (B.10) of [18].

Inserting the two expressions (3.2) and (3.3) in (3.1) we obtain a simple expression for the tachyon 3-point function

$$G_3^{\epsilon_1 \epsilon_2 \epsilon_3}(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{\pi^3 b^{\epsilon_1+\epsilon_2+\epsilon_3}} \lambda_L^{\frac{1}{b}(Q-\sum_i \alpha_i)} \lambda_M^{-\frac{1}{b}(e_0-\sum_i e_i)}, \quad (3.7)$$

reproducing an old result, see [19] and references therein. The partition function $Z_L(\lambda_L, \lambda_M, b)$ is conventionally determined identifying its third derivative with respect to λ_L with $-G_3^{+++}(b, b, b)$.

Apart from the power of λ_M , this expression does not depend on the presence of matter screening charges. On the other hand already the “neutrality” condition on the matter charges in (3.6), being simultaneously a constraint on the Liouville charges, simplifies drastically the constant (3.2) to (3.7).

The “matter-Liouville product” formula (3.7) is valid for generic momenta and under the normalization assumptions made for the two constants in the product. It is however formal, giving $0 \times \infty$, at the singular points of the constants. By the same reasons, the simple relation obtained by combining the matter and Liouville functional relations, cannot be expected to hold in general. Motivated by these observations we reconsider the problem of determining the tachyon 3-point function. It will be determined as the solution of a pair of difference equations which will be derived below as part of the set of functional identities for the n -point tachyon correlators. These equations are weaker than the combined matter plus Liouville functional identities and (3.7) is only the simplest of their solutions. For the normalized 3-point function N_{P_1, P_2, P_3} , defined by

$$\mathcal{G}^{\varepsilon_1 \varepsilon_2 \varepsilon_3}(P_1, P_2, P_3) = \frac{\lambda_L^{\frac{1}{2b}(\sum_i \varepsilon_i P_i - Q)} \lambda_M^{\frac{1}{2b}(e_0 - \sum_i P_i)}}{\pi^3 b^{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}} N_{P_1, P_2, P_3} \quad (3.8)$$

these equations read

$$N_{P_1 + b^\epsilon, P_2, P_3} + N_{P_1 - b^\epsilon, P_2, P_3} = N_{P_1, P_2 + b^\epsilon, P_3} + N_{P_1, P_2 - b^\epsilon, P_3}, \quad \epsilon = \pm 1. \quad (3.9)$$

We will see later that a possible solution of these equations is given by the $sl(2)$ “fusion rules”, i.e., the tensor product decomposition multiplicities, which take values 1 or 0. In the theory corresponding to the diagonal action (2.9) the factorization to matter \times Liouville does not hold and the 3-point function is determined by an equation of the same type, but with shifts of the momenta by e_0 .

4. The ground ring action on the tachyons

A crucial property of the operators a_\pm (2.6) is that their derivatives $\partial_z a_\pm$ and $\partial_{\bar{z}} a^\pm$ are BRST exact: $\partial_z a_- = \{Q_{\text{BRST}}, \mathbf{b}_{-1} a_-\}$. Therefore, any amplitude that involves a_\pm and other BRST invariant operators does not depend on the position of a_\pm . This property allows to write recurrence equations for the tachyon correlation functions using that the BRST invariant operators W_α^\pm form a module of the ground ring up to commutators with the BRST charge [3]

$$\begin{aligned} a_- W_\alpha^- &= -W_{\alpha - \frac{b}{2}}^-, & a_+ W_\alpha^+ &= -W_{\alpha - \frac{1}{2b}}^+ \\ a_- W_\alpha^+ &= a_+ W_\alpha^- = 0. \end{aligned} \quad (4.1)$$

Both relations follow from the free field OPE and are deformed in the presence of integrated tachyon vertex operators. Thus the second relation is modified to [4,5]

$$a_- W_\alpha^+ T_{\alpha_1}^+ = W_{\alpha+\alpha_1-\frac{b}{2}}^+, \quad a_+ W_\alpha^- T_{\alpha_1}^- = W_{\alpha+\alpha_1-\frac{1}{2b}}^-. \quad (4.2)$$

A particular example of (4.2) is provided by $P_1 = e_0$, i.e., $\alpha_1 = b$, or $\alpha_1 = 1/b$ respectively. In this case $T_{\alpha_1}^\pm$ coincide with the Liouville interaction terms in (2.7). Treating them as perturbations amounts to modifying the original ring generators as [6]

$$a_- \rightarrow a_- (1 - \lambda_L T_b^+ + \dots), \quad a_+ \rightarrow a_+ \left(1 - \tilde{\lambda}_L T_{1/b}^- + \dots\right) \quad (4.3)$$

Another deformation of the ring generators (2.6), involving the matter screening charges, corresponds to the choice $\alpha_1 = 0$, i.e., $P_1 = Q$ or $P_1 = -Q$ respectively. Furthermore the action of the ring generators is nontrivial on some particular double integrals

$$a_- W_\alpha^- T_{\alpha_1}^+ T_{b-\alpha_1}^+ = -W_{\alpha+\frac{b}{2}}^-, \quad a_+ W_\alpha^+ T_{\alpha_1}^- T_{\frac{1}{b}-\alpha_1}^- = -W_{\alpha+\frac{1}{2b}}^+. \quad (4.4)$$

The choices $\alpha_1 = b$ in the first and $\alpha_1 = \frac{1}{b}$ in the second relation in (4.4) correspond to the combined matter and Liouville first order perturbations.

Summarizing, the relations (4.1) get deformed as follows (we keep the same notation for the fully deformed ring generators):

$$\begin{aligned} a_- W_\alpha^+ &= -\lambda_L W_{\alpha+\frac{b}{2}}^+ - \lambda_M W_{\alpha-\frac{b}{2}}^+ \\ a_- W_\alpha^- &= -W_{\alpha-\frac{b}{2}}^- - \lambda_L \lambda_M W_{\alpha+\frac{b}{2}}^- \end{aligned} \quad (4.5)$$

$$\begin{aligned} a_+ W_\alpha^- &= -\tilde{\lambda}_L W_{\alpha+\frac{1}{2b}}^- - \tilde{\lambda}_M W_{\alpha-\frac{1}{2b}}^- \\ a_+ W_\alpha^+ &= -W_{\alpha-\frac{1}{2b}}^+ - \tilde{\lambda}_L \tilde{\lambda}_M W_{\alpha+\frac{1}{2b}}^+. \end{aligned} \quad (4.6)$$

The identities (4.5), (4.6) generalize the OPE relations obtained in [3,4,5,6]. The two terms in each of these relations are in fact the only one preserving the condition (2.4), out of the four terms in the OPE of the fundamental matter and Liouville vertex operators in (2.6). There are further generalizations of (4.4) with m integrals of tachyon operators of the same chirality, if the sum of the Liouville exponents is respectively $\frac{m}{2}b$ and $\frac{m}{2b}$. When all momenta correspond to screening charges, matter or Liouville, they appear in equal number $k = m/2$. The corresponding OPE coefficients vanish for $k > 1$, implying no new corrections to the two term action (4.5), (4.6) of the ring generators.

This is not the whole story, however, if each of the generators is deformed with all the four terms in (2.7). For some momenta on the lattice $kb + l/b$, $k, l \in \mathbb{Z}$ the OPE relations (4.5), (4.6) get further modified so that terms reversing the given chirality appear. In particular, restricting to tachyon momenta labelled by degenerate matter representations,

$$e_0 - 2e = P = \pm(-mb + n/b), \quad m, n \in \mathbb{Z}_{>0}, \quad (4.7)$$

the matter reflected images of the two terms in the r.h.s. also appear if both matter screening charges are taken into account along with one of the Liouville charges². Alternatively, one can “deform” the tachyon basis in the case of degenerate representations and consider combinations invariant under matter reflection.

For the operator a_+a_- perturbed by the diagonal interaction action (2.9) (to be denoted A_{+-}) we obtain a similar relation with shifts by $e_0/2$, which we shall write in terms of the momenta $P = \epsilon(Q - 2\alpha) = e_0 - 2e$:

$$\begin{aligned} A_{+-} \mathcal{W}_P^{(+)} &= \lambda_L \mathcal{W}_{P+e_0}^{(+)} + \tilde{\lambda}_L \lambda_M^{-\frac{e_0}{b}} \mathcal{W}_{P-e_0}^{(+)} \\ A_{+-} \mathcal{W}_\alpha^{(-)} &= \tilde{\lambda}_L \mathcal{W}_{P+e_0}^{(-)} + \lambda_L \lambda_M^{-\frac{e_0}{b}} \mathcal{W}_{P-e_0}^{(-)}. \end{aligned} \quad (4.8)$$

The diagonal action (2.9) is designed so that to project the four term action of the product a_-a_+ , deformed according to (4.5) and (4.6), to the two terms in (4.8). This relation holds true for generic momenta, as well as when being restricted to the diagonal $P = ke_0, k \in \mathbb{Z}$, which, with the exception of the point $P = 0$, describes the diagonal degenerate (order operator) fields.

5. Solutions of the functional equation for the 3-point function

Applying (4.5), (4.6) in a 4-point function with three tachyons we obtain the functional equations (3.9) for the tachyon 3-point functions. Analogous relation with shifts by e_0 follows from (4.8). We give here some examples solving these equations besides (3.7).

Restricting to tachyons labelled by degenerate matter representations (4.7) and setting $\lambda_M = 1$ we postulate that the tachyons labelled by the border lines for $m = 0$ or $n = 0$ vanish. Then one obtains as solutions of (3.9) the product of $sl(2)$ tensor product decomposition multiplicities of finite irreps of dimensions m_i, n_i ³

$$N_{P_1, P_2, P_3} = N_{m_1, m_2, m_3} N_{m'_1, m'_2, m'_3}, \quad P_i = -m_i b + m'_i \frac{1}{b}, \quad (5.1)$$

with

$$N_{m_1, m_2, m_3} = \begin{cases} 1 & \text{if } |m_1 - m_2| + 1 \leq m_3 \leq m_1 + m_2 - 1 \\ & \text{and } m_1 + m_2 + m_3 = \text{odd} \end{cases}; \quad (5.2)$$

These multiplicities satisfy the difference identity

$$N_{m_1+1, m_2, m_3} - N_{m_1-1, m_2, m_3} = N_{m_1+m_2, m_3, 1} - N_{m_1-m_2, m_3, 1}. \quad (5.3)$$

the r.h.s. of which indicates the deviation from the naive matter-Liouville product functional relation. In the diagonal case $m = n$ the equation implied by (4.8) is solved by one such factor identifying $P_i = \epsilon_i m_i e_0$.

² More precisely this holds for (4.7) taken with the plus sign.

³ In a recent derivation [2] of this result in the rational b^2 case instead of computing OPE coefficients the ground ring itself is identified with the fusion ring of the $c < 1$ minimal models.

A second example, now for $P \in \mathbb{R}$, is given by the expression dual to (5.1),

$$N_{P_1, P_2, P_3} = \sum_{m=0} \sum_{n=0} \left(2 \sin \pi m e_0 b \sin \pi n \frac{e_0}{b} \right)^2 \chi_{P_1}(m, n) \chi_{P_2}(m, n) \chi_{P_3}(m, n), \quad (5.4)$$

$$\chi_P(m, n) = \frac{\sin \pi m P b \sin \pi n P / b}{\sin \pi m e_0 b \sin \pi n e_0 / b} = \chi_{-P}(m, n)$$

in which the degenerate representations label the dual (boundary) variables.

A solution of the diagonal difference relations is given by the multiplicity projecting to diagonal matter charges

$$N_{P_1, P_2, P_3} = \sum_{l=0}^{\infty} \delta(P_1 + P_2 + P_3 - (2l + 1)e_0) \quad (5.5)$$

$$N_{P_1+e_0, P_2, P_3} - N_{P_1-e_0, P_2, P_3} = \delta(P_1 + P_2 + P_3).$$

There are analogous double sum solutions of the non-diagonal relations (3.9).

6. Functional equations for the 4-point tachyon amplitudes

The integrated tachyon vertex operators in the n -point functions, $n \geq 4$, play the role of screening charges and the OPE relations as (4.2) and (4.4) imply new channels in the action of the ring generators besides (4.5), (4.6). This leads to additional terms with less than n fields; alternatively these “contact” terms are expected to account for the skipped BRST commutators in the operator identities (4.5), (4.6). We write down as an illustration the relation for the 4-point function $G_4^{-+++}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \langle W_{\alpha_1}^- W_{\alpha_2}^+ T_{\alpha_3}^+ W_{\alpha_4}^+ \rangle$,

$$\begin{aligned} & G_4^{-+++}(\alpha_1 - \frac{b}{2}, \alpha_2, \alpha_3, \alpha_4) + \lambda_L \lambda_M G_4^{-+++}(\alpha_1 + \frac{b}{2}, \alpha_2, \alpha_3, \alpha_4) \\ & - \lambda_L G_4^{-+++}(\alpha_1, \alpha_2 + \frac{b}{2}, \alpha_3, \alpha_4) - \lambda_M G_4^{-+++}(\alpha_1, \alpha_2 - \frac{b}{2}, \alpha_3, \alpha_4) \\ & = -G_3^{-++}(\alpha_1, \alpha_2 + \alpha_3 - \frac{b}{2}, \alpha_4) \\ & + (\lambda_L \delta_{\alpha_3, 0} + \lambda_M \delta_{\alpha_3, b}) G_3^{-++}(\alpha_1 + \frac{b}{2}, \alpha_2, \alpha_4). \end{aligned} \quad (6.1)$$

The choice of which of the three chirality plus operators is represented by an integrated vertex should not be essential - this leads to a set of relations obtained from (6.1) by permutations of α_s , $s = 2, 3, 4$. The analogous to (6.1) identity for the function G_4^{+---} with reversed chiralities, resulting from (4.6), is obtained replacing $b \rightarrow 1/b$, $\lambda_L \rightarrow \tilde{\lambda}_L$, $\lambda_M \rightarrow \tilde{\lambda}_M$. Similar relations, but with shifts $e_0/2$, are obtained from the diagonal ring action (4.8).

The relations for the 4-point function are expected to hold for generic tachyon momenta. Besides the two special contact terms in the last line in (6.1), which come from the double integral relation (4.4), there are potentially more terms in the quasi-rational case, which correspond to multiple integrals generalizing (4.4). These integrals are not of the type in [18] and the missing information on these contact terms is a main problem. Furthermore the functional identity (6.1) and its dual correspond each to one of the interaction

actions in (2.7) and they apply to a restricted class of correlators. Taking into account the full deformation as discussed above adds new potential contact terms, including ones for generic momenta which modify the recurrence identities.

In the absence of matter screening charges, $\sum_i e_i - e_0 = 0$, the solutions of the partially deformed ring relation obtained setting $\lambda_M = 0$ in (6.1), reproduce the 4-point functions found by other means in [19]. Furthermore the functional relations admit solutions with fixed number of matter screening charges $\sum_i e_i - e_0 = mb - n/b$. Another application of these relations is the case of one degenerate and three generic fields. As an example we take $P_2 = (\frac{1}{b} - (m+1)b)$, $m \in \mathbb{Z}_{\geq 0}$ in the matter degenerate range, imposing the vanishing of the tachyon at the border value $P_1 = \frac{1}{b}$ of (4.7). The equation (6.1) is solved recursively, taking as initial condition $G_4^{-+++}(\alpha_1, b, \alpha_3, \alpha_4) = -\partial_{\lambda_L} G_3^{-++}(\alpha_1, \alpha_3, \alpha_4)$. Skipping the overall normalization the result reads

$$\begin{aligned} \hat{G}_4^{-+++}(\alpha_1, a_2 = b + \frac{mb}{2}, \alpha_3, \alpha_4) &= (m+1) \left(\sum_{s \neq 1} \alpha_s - Q + \frac{mb}{2} \right) = (m+1) \left(\sum_{i=1}^4 \alpha_i - Q - b \right) \\ &= (m+1) \left(\frac{Q}{2} + \frac{mb}{2} \right) - \frac{1}{2} \sum_{s \neq 2} \sum_{k_s=0}^m \epsilon_s (P_s - mb + 2k_s b) \end{aligned} \quad (6.2)$$

and there is an analogous formula for the dual correlator $G_4^{+---}(a_1, \alpha_2 = \frac{1}{b} + \frac{n}{2b}, \alpha_3, \alpha_4)$.⁴

The situation with the contact terms is simpler in the theory described by the action (2.9). Consider the case of diagonal $m = n$ degenerate matter representations with 3-point function given by the multiplicity (5.2). The solution for the 4-point function has the structure of a three channel expansion, generalizing a formula of [19] in the case of trivial matter. It reads, skipping the universal λ_L, λ_M prefactor

$$\begin{aligned} G_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= -\frac{1}{2\pi^3 b e_0} \left(\sum_{m=0} (N_{m_1, m_2, m} (\frac{Q}{3} - m e_0) N_{m, m_3, m_4} + \text{permutations}) \right) \\ &= \frac{1}{\pi^3 b e_0} N_{m_1, m_2, m_3, m_4} \left(Q + b - \sum_i \alpha_i - \frac{1}{2} e_0 (N_{m_1, m_2, m_3, m_4} - 1) \right), \quad \alpha_i = \frac{Q}{2} - m_i \frac{e_0}{2}. \end{aligned} \quad (6.3)$$

Assuming that the largest of the values m_i is say, m_1 , i.e., $m_1 \geq m_s, s = 2, 3, 4$, the symmetric in the four arguments function (6.3) can be identified with a correlator of type G_4^{-+++} . The contact term is given by a linear combination of the 3-point functions

$$[N]_{m_1, m_2+m_3, m_4} := N_{m_1, m_2+m_3, m_4} - N_{m_1, |m_2-m_3|, m_4}, \quad (6.4)$$

which takes the values $0, \pm 1$. The second term in (6.4) reflects the fact that the fields vanish at the border $m = 0$ of the diagonal of (4.7) and can be interpreted as antisymmetric in P combinations so that a tachyon and its matter (plus Liouville) reflection image are identified.

⁴ From these fixed chirality formulae one can extract a symmetric in the momenta (“local”) correlator, generalizing the non-analytic expression of [19], with physical intermediate momenta in each of the three channels.

7. Microscopic realization of the diagonal CFT

7.1. The SOS model as a discretization of the gaussian matter field

If the matter field is a free gaussian field with a background charge, *i.e.* when $\lambda_M = 0$, the 2D gravity can be realized microscopically as a particular solid-on solid (SOS) model with complex Boltzmann weights [20]. The local fluctuation variable in the SOS model is an integer *height* $x \in \mathbb{Z}$ and the acceptable height configurations are such that the heights of two nearest-neighbor points are either equal or differ by ± 1 . Therefore each SOS configuration defines a set of domains of equal height covering the two-dimensional lattice. The boundaries of the domains form a pattern of non-intersecting loops on the lattice. In this way the SOS model is also described as a *loop gas*, *i.e.* as an ensemble of self-avoiding and mutually avoiding loops, which rise as the boundaries between the domains of equal height.

The loop gas describes a whole class of solvable height models, as the restricted SOS models (RSOS) [21] and their *ADE* generalizations [22], in which the target space \mathbb{T} , in which the height variable takes its values, is the ensemble of the nodes of a simply-laced (*ADE*-type) Dynkin graph. The local Boltzmann weights of the height models depend on the “mass” of the loops M and on the components $S = \{S_x\}_{x \in \mathbb{T}}$ of an eigenvector of the adjacency matrix $A = \{A_{x,x'}\}$ of the graph \mathbb{T} :

$$\sum_{x'} A_{x,x'} S_{x'} = 2 \cos(\pi p_0) S_x. \quad (7.1)$$

For the unitary *ADE* models this is the Peron-Frobenius (PF) vector.

The weight of each height configuration factorizes to a product of the weights of the connected domains and the loops representing the domain boundaries. The weight of a domain \mathcal{D} is

$$\Omega_{\mathcal{D}}(x) = (S_x)^{2-n}, \quad n = \# \text{ boundaries of } \mathcal{D}. \quad (7.2)$$

In addition, the loops, or the domain boundaries, are weighted by a factor $\exp(-M \times \text{Length})$, similarly to the droplets in the Ising model. The sum over heights can be easily performed using the relation (7.1) and the result is that each loop acquires a factor $2 \cos(\pi p_0)$.

In the SOS model, the role of the PF vector S is played by

$$S_x = \frac{1}{\sqrt{2}} e^{i\pi p_0 x} \quad (x \in \mathbb{Z}) \quad (7.3)$$

where p_0 is any real number in the interval $[0, 1]$. It has been conjectured [10] using some earlier arguments of [23], that the critical behavior of the SOS-model on a lattice with curvature defects is described in the continuum limit by a gaussian field χ theory with action (2.1). The height variable x and the background momentum p_0 are related to the gaussian field χ and the background electric charge $e_0 = \frac{1}{b} - b$ as [24]⁵

$$p_0 = \frac{1}{b^2} - 1, \quad \chi = \pi x/b. \quad (7.4)$$

⁵ Here we are considering only the dilute phase of the loop gas. The correspondence in the dense phase is $p_0 = 1 - b^2$, $\chi = \pi b x$.

If the SOS model is considered in the ensemble of planar graphs with given topology, its continuum limit will be described by the full action (2.1). The Liouville coupling in (2.7) is controlled by an extra factor $e^{-\lambda_L \times \text{Area}}$ in the Boltzmann weights (7.2). In the dilute phase $\lambda_L \sim M^2$. The detailed description of the SOS model coupled to gravity can be found in [10].

7.2. The theory with $\lambda_M \neq 0$ as a semi-restricted SOS model (SRSOS)

One can argue that the microscopic realization of the theory (2.1) deformed by the term (2.9) is given for generic b by a “semi-restricted” height model coupled to gravity, with target space $\mathbb{T} = \mathbb{Z}_{>0}$. The Boltzmann weights are defined by (7.2), with

$$S_x = \sqrt{2} \sin(\pi p_0 x) \quad (x \in \mathbb{Z}_{>0}). \quad (7.5)$$

The background charge and the normalization of the field are again given by (7.4), for all values of b . We will see later that, with this identification, the four point functions in the ‘diagonal’ CFT (2.9) and the SRSOS loop model coincide up to a numerical factor.

The order local operators in the SOS and SRSOS models are constructed by inserting the wave functions

$$\psi_p(x) = e^{i\pi(p-p_0)x} \quad \text{for SOS}, \quad \psi_p(x) = \frac{\sin(\pi p x)}{\sin(\pi p_0 x)} \quad \text{for SRSOS}. \quad (7.6)$$

In the continuum limit the operators (7.6) are described by conformal fields with dimensions $\Delta_p = \frac{p^2 - p_0^2}{4(1+p_0)} = \frac{(P^2 - e_0^2)}{4}$.

7.3. Target space diagram technique

The loop gas representation allows to build a target space diagram technique for the string path integral, described in [10,25,12]. The n -point functions are given by the sum of all possible Feynman diagrams composed by vertices, propagators, tadpoles and leg factors. The rules are that the vertices can be attached either to tadpoles, or to the propagators, or to leg factors. It is forbidden to attach directly two vertices, or propagator with a tadpole. We summarize the Feynman rules in Fig. 1, where we drew the extremities of the lines in such a way that the rules for gluing them come naturally. All the elements of the diagram technique depend on two types of quantum numbers: the periodic target space momentum $p + 2 \equiv p$ and the nonnegative integers k .

- Propagator $D_{k,k'}(p)$

$$\begin{aligned} D_{00}(p) &= -(|p| - \tfrac{1}{2})(|p| - \tfrac{3}{2}) \\ D_{01}(p) &= -\tfrac{1}{2}(p^2 - \tfrac{1}{4})(|p| - \tfrac{3}{2})(|p| - \tfrac{5}{2}) = D_{10}(p), \\ D_{11}(p) &= -\tfrac{1}{2}(p^2 - \tfrac{1}{4})(|p| - \tfrac{3}{2})(|p| - \tfrac{5}{2}) \left[1 + \tfrac{1}{3}(|p| - \tfrac{5}{2})(|p| - \tfrac{7}{2})\right], \quad \text{etc.} \end{aligned} \quad (7.7)$$

- Vertices $V_{k_1, \dots, k_n}(p_1, \dots, p_n)$:

$$V_{k_1, \dots, k_n}(p_1, \dots, p_n) = \frac{(k_1 \dots + k_n)!}{k_1! \dots k_n!} N_{p_1, \dots, p_n},$$

$$\text{where } N_{p_1, \dots, p_n} = \sum_{x \in \mathbb{N}} S_x^2 \prod_{j=1}^n \psi_{p_j}(x) \quad (7.8)$$

- External line factors $\Gamma_k(p)$:

$$\Gamma_k(p) = \frac{1}{k!} \left(\frac{1}{2} + p\right)_k \left(\frac{1}{2} - p\right)_k \quad (7.9)$$

- Tadpole $B_k(p)$:

$$B_0(p) = B_1(p) = 0, \quad B_k(p) = -\delta(p, p_0) \Gamma_{k-1}(g), \quad k = 2, 3, \dots \quad (7.10)$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol. The propagator, vertices and tadpole are defined above for the interval $-1 < p < 1$. Their definition is extended to arbitrary values of p by requiring periodicity $p \rightarrow p + 2$. The leg factors are not required to be periodic.

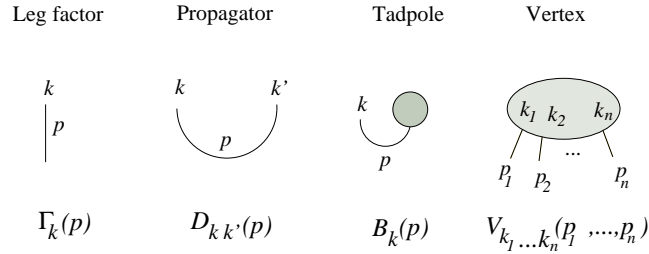


Fig.1 : Feynman rules for the correlation functions

7.4. General formula for the 4-point function

The 4-point function is given by the sum of the three Feynman diagrams shown in Fig. 2. Each Feynman diagram stands for the sum of terms that differ by permutations of the external legs.

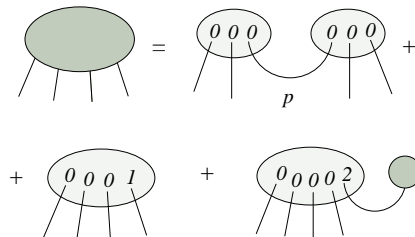


Fig.2: The diagrams for the 4-point function

The corresponding analytic expression is

$$\begin{aligned}
G(p_1, p_2, p_3, p_4) = & \int_{-1}^1 dp [\Gamma_0(p_1)\Gamma_0(p_2)N_{p_1 p_2 p} D_{00}(p)N_{-pp_3 p_4}\Gamma_0(p_3)\Gamma_0(p_4) + \text{permutations}] \\
& + \left[\Gamma_0(p_1)\Gamma_0(p_2)\Gamma_0(p_3)\Gamma_1(p_4) + \text{permutations} \right] N_{p_1 p_2 p_3 p_4} \\
& + B_2(p_0)\Gamma_0(p_1)\Gamma_0(p_2)\Gamma_0(p_3)\Gamma_0(p_4)N_{p_0 p_1 p_2 p_3 p_4}
\end{aligned} \tag{7.11}$$

Then, using (7.9)-(7.7) we rewrite (7.11) as

$$\begin{aligned}
G(p_1, p_2, p_3, p_4) = & \left[(1 \pm p_0)^2 - \frac{1}{4} + \sum_{s=1}^4 \left(\frac{1}{4} - p_s^2 \right) \right] N_{p_1, p_2, p_3, p_4} \\
& + \int_{-1}^1 dp (N_{p_1, p_2, p} N_{-p, p_3, p_4} + N_{p_1, p_3, p} N_{-p, p_2, p_4} + N_{p_1, p_4, p} N_{-p, p_2, p_3}) \left(|p| - \frac{1}{2} \right) \left(|p| - \frac{3}{2} \right)
\end{aligned} \tag{7.12}$$

This formula is valid for both height models, SOS and SRSOS; in the second case the integration runs over half of the interval. It is true also for the *ADE* rational string theories, if the integral over p is replaced by the corresponding discrete sum. In the case of degenerate fields we consider the order operators (7.6) with $p = mp_0$, $m = 1, 2, 3, \dots$. The integral in (7.12) is replaced by a summation over the positive integers. In the 3-point multiplicity in (7.7) the summation is replaced by an integral, $xp_0 \in [0, 2]$, i.e., the multiplicity is given by the standard integral representation of (5.2). Returning to the normalizations and notation used in the worldsheet theory, $P_i = \varepsilon_i b p_i = \varepsilon_i m_i b p_0$, $e_0 = b p_0$, we recover precisely the 4-point formula (6.3).

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References

- [1] E. Witten, Nucl. Phys. B 373 (1992) 187, hep-th/9108004.
- [2] N. Seiberg and D. Shih, JHEP 0402 (2004) 021, hep-th/0312170.
- [3] D. Kutasov, E. Martinec and N. Seiberg, Phys. Lett. B 276 (1992) 437, hep-th/9111048.
- [4] M. Bershadsky and D. Kutasov, Nucl. Phys. B 382 (1992) 213, hep-th/9204049.
- [5] S. Kachru, Mod. Phys. Lett. A7 (1992) 1419, hep-th/9201072.
- [6] I.K. Kostov, Nucl. Phys. B 689 (2004) 3, hep-th/0312301.
- [7] I.K. Kostov and V.B. Petkova, Non-Rational 2D Quantum gravity: Ground Ring versus Loop Gas, in preparation.
- [8] Al.B. Zamolodchikov, On the Three-point Function in Minimal Liouville Gravity, hep-th/0505063.
- [9] I.K. Kostov, Nucl. Phys. B326 (1989) 583.
- [10] I.K. Kostov, Nucl. Phys. B 376 (1992) 539, hep-th/9112059.
- [11] I.K. Kostov and M. Staudacher, Phys. Lett. B 305 (1993) 43, hep-th/9208042.
- [12] S. Higuchi and I.K. Kostov, Phys. Lett. B 357 (1995) 62, hep-th/9506022.
- [13] F. David, Mod. Phys. Lett. A3 (1988) 1651.
- [14] J. Distler and H. Kawai, Nucl. Phys. B 321 (1989) 509.
- [15] H. Dorn and H.J. Otto: Nucl. Phys. B 429 (1994) 375, hep-th/9403141.
- [16] A.B. Zamolodchikov and Al.B. Zamolodchikov, Nucl. Phys. B 477 (1996) 577, hep-th/9506136.
- [17] J. Teschner, Phys. Lett. B 363 (1995) 65, hep-th/9507109.
- [18] Vl.S. Dotsenko and V.A. Fateev, Nucl. Phys. B 251 [FS13] (1985) 691.
- [19] P. Di Francesco and D. Kutasov, Nucl. Phys. B 375 (1992) 119, hep-th/9109005.
- [20] B. Nienhuis, “Coulomb gas formulation of 2-d phase transitions”, *in* Phase transitions and critical phenomena, Vol. 11, ed. C.C. Domb and J.L. Lebowitz (Academic Press, New York, 1987), ch.1.
- [21] G. E. Andrews, R. J. Baxter and P. J. Forrester, J. Stat. Phys. 35 (1984) 193.
- [22] V. Pasquier, Nucl. Phys. B 285 (1987) 162.
- [23] O. Foda and B. Nienhuis, Nucl. Phys. B 324 (1989) 643.
- [24] I.K. Kostov, B. Ponsot and D. Serban, Nucl. Phys. B 683 (2004) 309, hep-th/0307189.
- [25] V.A. Kazakov and I.K. Kostov, Nucl. Phys. B 386 (1992) 520, hep-th/9205059.